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On the Approximation of the Linear Equations with Bounded Kernels

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The existence of some projection-like approximation method is proved for the linear equations with bounded kernels. The proposed method does not imply that the kernel must be a compact one, or the invertible operator is an A-proper one. This fact permits the solution of linear equations, which cannot be solved by the Garlekin method.

1. INTRODUCTION

In this paper we shall be concerned with the approximate solution of the equation

$$Ax = y \tag{1.1}$$

where A is a linear bounded operator defined on the (complex) Hilbert space E, which has a Schauder basis. We assume also that A^{-1} exists, therefore Eq. (1.1) has only one solution. One special form of the general problem is the solution of the equation

$$x - Kx = y \tag{1.2}$$

where K is the kernel of the equation.

As is well known, different approximation methods exist to solve Eqs. (1.1) or (1.2), and a general theory can be found in [1]. The common thing in these approaches is the basic assumption that K is a compact or weakly compact [2] operator. The applicability of the Garlekin methods [3] can be generalized when more complicated requirements are used as collectively compact projection-like approximations of the kernel K as in [4] or A-proper mappings in [5].

In this paper we shall prove that if the space E is separable and possesses a Schauder basis, A is a bounded operator, and A^{-1} exists then the

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problem (1.1) can be solved by a projection-like method, e.g., by a successive approximation by finite-dimensional matrices. In this sense this paper is a generalization of the work done in Ref. [6], where the case of the generalization of the Fredholm alternative for integral equations with bounded kernels was studied.

The requirement of the boundedness of the operator A is simpler and can be checked easier in the practical applications. As an example, in Section 3 we study the well known case of an operator which is not A-proper but it is bounded and A^{-1} exists, for which the A-proper projection method cannot be applied but the proposed method can solve Eq. (1.1). In many physical problems, the N-body problem can be described off the energy shell by the N-body Lippmann–Schwinger equation, which corresponds to an integral equation with a non-compact but simply bounded kernel [7]. The usual restriction imposed by the compactness of the kernel for the solvability of the integral equations as in (1.2) imposes complicated formulations [7–9] even for the off-shell N-body problems. An alternative theory of integral equations assuming only the boundedness of the kernel might provide a simplification for this kind of probem.

In Section 2, the general theory of inversion for bounded operators is given. In Section 3 an algorithm is proposed for the solution of Eq. (1.1). This algorithm is applied in a case of bounded invertible operator which is not A-proper and the effectiveness of the proposed method is discussed.

2. General Theory

Let E be a complex Hilbert space with an orthonormalized Schauder basis $\{e_n\}$, E_n is the linear subspace of E such that

$$E_n = \operatorname{span}\{e_1, e_2, ..., e_n\}, \tag{2.1}$$

P_n is the orthogonal projection on E_n, and Q_n is the projection on the orthogonal complement of E_n.

The operator A is assumed to be bounded and A^{-1} exists. We define

$$A_n = P_n A \tag{2.2}$$

and it is trivial to see that

$$A_n = \sum_{k=1}^{n} e_k f_k^* \tag{2.3}$$

where f_n^* is the linear functional on E, which corresponds to the element f_n defined as

$$f_n = A^* e_n \tag{2.4}$$

where A^* is the adjoint of the operator A. If the operator A is bounded then the sequence of operators $\{A_n\}$ converges strongly to A, i.e.,

$$A_n \xrightarrow{s} A.$$

If the operator A^{-1} exists then the elements $\{f_1, f_2, \ldots\}$ defined by (2.4) *are linearly independent.*

This statement is true because if

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

then from (2.4) we have

$$A^*(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) = 0.$$

The operator A^* has an inverse $(A^{-1})^*$ therefore

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0$$

which is not true, unless $c_1 = c_2 = \cdots = c_n = 0$. Let $\tilde{E}_n$ be the linear subspace of E such that

$$\tilde{E}_n = \operatorname{span}\{f_1, f_2, \ldots, f_n\},$$

$\tilde{P}_n$ is the corresponding projection, and $\tilde{Q}_n$ the projection on the orthogonal complement.

If A^{-1} exists then $\tilde{P}_n \to^s I$, where I is the unit in the space E. This statement is true because the sequence of operators $\{\tilde{P}_n\}$ is an increasing bounded sequence of positive definite operators therefore the sequence converges strongly to some projection operator P. If g is an element of the space $(I-P)\,E$ then by definition

$$Ag = 0$$

but the existence of A^{-1} implies that $g = 0$, therefore $P = 0$.

We define the operator C_n as

$$C_n = A_n^* A_n.$$

If $\{e_n\}$ is an orthonormalized basis in E, then

$$C_n = \sum_{k=1}^{n} f_k f_k^*. \tag{2.5}$$

The sequence $\{C_n\}$ is an increasing sequence of positive operators. The following lemma was proved in Ref. [6].

LEMMA 1. *If $\{C_n\}$ is an increasing sequence of positive definte operators, then the sequence $\{G_n(\mu)\}$ where*

$$G_n(\mu)=(I+C_n)^{-1}[I-\mu(I+C_n)^{-1}]^{-1} \tag{2.6}$$

for $0\leqslant\mu<1$ is a decreasing sequence of positive definite operators.

Equation (2.6) can be written in a simpler form:

$$G_n(\mu)=[C_n+(1-\mu)I]^{-1}. \tag{2.7}$$

If C_n is the operator defined by Eq. (2.5) we can see that the subspace $\tilde{E}_n$ is an invariant subspace of C_n, therefore

$$G_n(\mu)=H_n(\mu)+\tilde{Q}_n/(1-\mu) \tag{2.8}$$

where $H_n(\mu)$ is an operator acting on $\tilde{E}_n$ only. The exact form of $H_n(\mu)$ in $\tilde{E}_n$ is

$$H_n(\mu)=[C_n+(1-\mu)\tilde{I}_n]^{-1}$$

i.e., is the inverse of $C_n+(1-\mu)\tilde{I}_n$ in the space $\tilde{E}_n$ where $\tilde{I}_n$ is the unit operator in this space. For $\mu=1$ the operator $H_n(1)$ is well defined in $\tilde{E}_n$ because the C_n^{-1} exists in $\tilde{E}_n$. Consequently the singularity of the operator $G_n(\mu)$ given by Eq. (2.8) is due to the part $\tilde{Q}_n/(1-\mu)$.

A direct application of Lemma 1 is

$$\|G_{n+1}(\mu)\|<\|G_n(\mu)\|$$

but

$$\|G_n(\mu)\|=\|H_n(\mu)\|+\|\tilde{Q}_n\|/(1-\mu)=\|H_n(\mu)\|+1/(1-\mu).$$

Consequently

$$\|H_{n+1}(\mu)\|<\|H_n(\mu)\| \tag{2.9}$$

for every $0\leqslant\mu<1$. This inequality implies that the well defined operator $D_n=H_n(1)$ satisfies the inequality

$$\|D_{n+1}\|<\|D_n\|. \tag{2.10}$$

It is trivial to see that

$$D_n - C_n = \tilde{P}_n \tag{2.11}$$

because $\tilde{P}_n$ is the unit operator $\tilde{I}_n$.

The defintion (2.5) of the operator C_n implies that

$$C_n \xrightarrow{s} C = A^*A. \tag{2.12}$$

Consequently

$$D_n \xrightarrow{s} (A^*A)^{-1} = C^{-1} \tag{2.13}$$

because for every f in E we have

$$(D_n - C^{-1})f = (D_n C - I)\, C^{-1}f = D_n(C - C_n)\, C^{-1}f - \tilde{Q}_n C^{-1}f$$

then

$$\|(D_n - C^{-1})f\| < \|D_n\| \cdot \|(C - C_n)\, C^{-1}f\| + \|\tilde{Q}_n C^{-1}f\|. \tag{2.14}$$

Inequality (2.10) means that $\|D_n\|$ is uniformly bounded. From (2.12) and the fact that

$$\tilde{Q}_n \xrightarrow{s} 0$$

we conclude that (2.13) is true.

The direct result of (2.13) is that

$$D_n A_n^* \xrightarrow{s} A^{-1}$$

and we have shown that:

THEOREM. *If E is a separable Hilbert space with a Schauder basis and A is a linear bounded operator, such that A^{-1} exists, then if*

$$A_n = P_n A$$

we can construct an operator D_n such that

$$D_n(A_n^* A_n) = \tilde{P}_n$$

and

$$D_n A_n^* \xrightarrow{s} A^{-1}$$

(*The projections P_n and $\tilde{P}_n$ are defined as above.*)

3. Algorithm for the Construction of the Solution

In this section we will calculate the approximate solutions of the equation

$$Ag = h. \tag{3.1}$$

An approximate equation is defined as

$$A_n g^{(n)} = Pn\, h = h^{(n)} \tag{3.2}$$

where

$$A_n = P_n A$$

and

$$g^{(n)} = \tilde{P}_n g. \tag{3.3}$$

The Theorem of the preceding section implies that some operator F_n exists such that

$$F_n A_n = \tilde{P}_n \tag{3.4}$$

and

$$F_n \xrightarrow{s} A^{-1}.$$

Equations (3.2) and (3.4) imply

$$g^{(n)} = F_n h^{(n)} = F_n P_n h$$

and

$$g^{(n)} \rightarrow g.$$

Consequently Eq. (3.2) is an approximate equation to the original one (3.1).

Let

$$P_n h = h^{(n)} = \sum_{k=1}^{n} e_k (e_k, h) = \sum_{k=1}^{n} e_k \cdot h_k.$$

Equation (3.2) can be written as

$$(e_k, A_n g^{(n)}) = h_k = (e_k, h) \qquad \text{for } k = 1, 2, ..., n.$$

The operator A_n is given by Eqs. (3.2)–(3.3), so the last equation can be written as

$$(f_k, g^{(n)}) = h_k, \qquad k = 1, 2, ..., n. \tag{3.5}$$

It is known [1] that we can choose a basis $\{x_1, x_2, ..., x_4\}$ for the space $\tilde{E}_n$ as follows:

$$x_n = x'_n / \|x'_n\|$$

and

$$x'_n = \begin{vmatrix} (f_1, f_1) & (f_1, f_2) & \cdots & (f_1, f_{n-1}) & f_1 \\ (f_2, f_1) & (f_2, f_2) & \cdots & (f_2, f_{n-1}) & f_2 \\ & & --- & & \\ (f_n, f_1) & (f_n, f_2) & \cdots & (f_n, f_{n-1}) & f_n \end{vmatrix} \tag{3.6}$$

and

$$(x'_n, f_k) = 0 \qquad \text{if } k < n$$

$$(x'_n, f_n) = \Delta_n \qquad \text{where } \Delta_n \text{ is the Gramm determinant } \det\{(f_i, f)\}$$

and

$$\|x'_n\| = \sqrt{\Delta_{n-1} \Delta_n}$$

where

$$\Delta_0 \equiv 1.$$

If

$$g = \sum_{k=1}^{\infty} x_k \tilde{g}_k$$

the system of Eqs. (3.5) can be written in a matrix form as

$$\begin{Bmatrix} (f_1, x_1) & 0 & 0 & \cdots & 0 \\ (f_2, x_1) & (f_2, x_2) & 0 & \cdots & 0 \\ (f_3, x_1) & (f_3, x_2) & (f_3, x_3) & \cdots & 0 \\ & & --- & & \\ (f_n, x_1) & (f_n, x_2) & (f_n, x_3) & \cdots & (f_n, x_n) \end{Bmatrix} \begin{Bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \\ -- \\ \tilde{g}_n \end{Bmatrix} = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ -- \\ h_n \end{Bmatrix} \tag{3.7}$$

which is Eq. (3.2) under a matrix form.

The system (3.8) can be solved numerically very easily because

$$\tilde{g}_1 = h_1/(f_1, x_1)$$

and

$$\tilde{g}_k = \left(h_k - \sum_{l=1}^{k} (f_k, x_l)\,\tilde{g}_l\right) \Big/ (f_k, x_k) \tag{3.8}$$

and

$$(f_k, x_k) = \sqrt{\frac{\Delta_k}{\Delta_{k-1}}}$$

and

$$g^{(n)} = \tilde{P}_n g = \sum_{k=1}^{n} x_k \tilde{g}_k.$$

After a little algebra the coefficients $\tilde{g}_n$ can be written as

$$\tilde{g}_n = \frac{1}{\sqrt{\Delta_n \Delta_{n-1}}} \begin{vmatrix} (f_1,f_1)\,(f_1,f_2) \cdots (f_1,f_{n-1})\,h_1 \\ (f_2,f_1)\,(f_2,f_2) \cdots (f_2,f_{n-1})\,h_2 \\ - - - \\ (f_n,f_1)\,(f_n,f_2) \cdots (f_n,f_{n-1})\,h_n \end{vmatrix}. \tag{3.9}$$

The effectiveness of the proposed method is given by the obvious equality

$$\| g - g^{(n)} \| = \| \tilde{Q}_n A^{-1} h \|. \tag{3.10}$$

So a suitable choice of the basis $\{e_n\}$ for a given h can reduce considerably the rate of convergence, although there does not exist a uniform criterion of convergence which does not depend on the element h, as is expected because even for the operator A the convergence of the approximation is the strong one.

One obvious application of the method described above is that the usual Fredholm integral equations of second kind can be solved using a finite-dimensional matrices approximation if the kernel of the equation is a bounded operator in a separable Hilbert space with an orthonormalized basis. This result can also be generalized to the linear Garlekin-type equations [1], and in this case the compactness of the kernel is not a necessary restriction but only the boundness of the kernel is required.

One interesting application of the proposed method is the example of bounded operators which are not A-proper but have an inverse [5]. For this kind of operator Petryshyn's method cannot be applied, but there is not any difficulty in applying the proposed method. For this purpose Example 1.1a of [5] is examined.

Let $\{\cdots \varphi_{-3}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \varphi_1, \varphi_2, \cdots\}$ be a complete orthonormalised basis of the Hilbert space E, T is the bounded linear operator given by $T\varphi_i = \varphi_{i-1}$ for $i = 0, 1, 2, \ldots$. This operator is not A-proper but the inverse operator exists and is defined as $T^{-1}\varphi_i = \varphi_{i+1}$ for $i = 0, \pm 1, \pm 2, \ldots$. We define the basis $\{e_1, e_2, e_3, \ldots\}$ as

$$e_1 = 0, \qquad e_{2m} = \varphi_m, \qquad e_{2m+1} = \varphi_{-m}, \qquad m = 1, 2, \ldots.$$

So the elements f_n defined by (2.4) are given as

$$f_{2m} = Te_{2m} = e_{2(m+1)}, \quad m = 2, 3, 4, \ldots$$

$$f_{2m+1} = Te_{2m+1} = e_{2m+3}, \quad m = 0, 1, 2, \ldots$$

$$f_2 = Te_2 = e_1.$$

The new basis defined by (3.6) x'_n is easily calculated to be $x'_m = f_n$ because $\langle f_m, f_n \rangle = \delta mn$. If we have to solve the problem

$$Tg = h \tag{3.11}$$

the $\tilde{g}_n$ given by (3.9) are calculated to be

$$\tilde{g}_n = h_n.$$

Therefore

$$g = T^{-1}h = \sum \tilde{g}_n x'_n = h_1 e_3 + h_2 e_1 + h_3 e_5 + h_4 e_2 + \cdots$$

Consequently we have seen the application of the proposed method in one example that does not use an A-proper bounded operator.

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